

# Geometry of $\mathrm{GL}_n(\mathbb{C})$ on infinity: hinges, complete collineations, projective compactifications, and universal boundary

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Compactifications of semisimple groups and of homogeneous spaces arise in natural ways in different branches of mathematics (enumerative algebraic geometry, noncommutative harmonic analysis, automorphic forms etc.). The most important construction of this kind is the family of objects that are called the *Satake boundary*, or the *De Concini–Procesi boundary* or the *wonderful compactification*, see [26], [7], [4], [12], see also [22]. For the first time, such compactifications of the group  $\mathrm{PGL}(n, \mathbb{C})$  (the *complete collineations*) and of the symmetric space  $\mathrm{PGL}(n, \mathbb{C})/\mathrm{PO}(n, \mathbb{C})$  (the *complete quadrics*) were discovered by Semple [27], [28], [29]<sup>2</sup>.

In the geometry of symmetric spaces and in the analysis on symmetric spaces, there arise some more complicated compactifications as the Karpelevich compactification ([11], [15], [8]) and the Martin compactification ([6], [23], [8]).

There exists also a wide theory of algebraic equivariant compactifications of reductive groups and their homogeneous spaces, see [30], [16], [12], [24], [2] (if a reductive group is a torus, then this theory becomes the theory of toric varieties).

This paper has two purposes. The first aim is to give an explicit description in elementary geometric terms of all the algebraic projective compactifications (see below) of  $\mathrm{GL}_n(\mathbb{C})$ . The second aim is to construct a universal object (the sea urchin) for all the compactifications of this type.

**0.1. Projective compactifications.** Consider the group  $\mathrm{GL}_n(\mathbb{C})$  of all complex invertible  $n \times n$  matrices. Consider a polynomial (generally speaking, reducible) representation  $\zeta$  of the group  $\mathrm{GL}_n(\mathbb{C})$  in an  $N$ -dimensional complex linear space  $Z$ .

Denote by  $\mathrm{Mat}(Z)$  the space of all linear operators in  $Z$ . Consider the space  $\mathbb{P}\mathrm{Mat}(Z)$  consisting of nonzero operators defined up to a nonzero scalar factor. This space is the complex projective space  $\mathbb{CP}^{N^2-1}$ . Consider the maps

$$\mathrm{GL}_n(\mathbb{C}) \xrightarrow{\zeta} \mathrm{Mat}(Z) \rightarrow \mathbb{P}\mathrm{Mat}(Z).$$

Denote by  $[\mathrm{GL}_n]_{\zeta}$  the closure of the image of  $\mathrm{GL}_n(\mathbb{C})$  in  $\mathbb{P}\mathrm{Mat}(Z)$ . The spaces  $[\mathrm{GL}_n]_{\zeta}$  are called *projective compactifications* of  $\mathrm{GL}_n(\mathbb{C})$ , see [25].

REMARK. Let  $\zeta$  be an irreducible representation with a signature  $\nu = (\nu_1, \dots, \nu_n)$  (see below 2.12) satisfying the condition

$$\nu_1 > \nu_2 > \dots > \nu_n.$$

In this case, the space  $[\mathrm{GL}_n]_{\zeta}$  is called the Semple complete collineation space or the De Concini–Procesi compactification of  $\mathrm{PGL}_n(\mathbb{C})$ .  $\square$

**0.2. Abstract characterization of projective compactifications.** Let the group  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$  act on a projective algebraic variety  $M$ . Denote by  $\Delta \simeq \mathrm{GL}(n, \mathbb{C})$  the diagonal subgroup in  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ . Let  $G$  have an open  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ -orbit on  $M$ , and the stabilizer of the orbit contain  $\Delta$  (any proper subgroup of  $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$  containing  $\Delta$  is a product of  $\Delta$  and a subgroup in the center  $\mathbb{C}^* \times \mathbb{C}^*$  of  $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ ).

<sup>1</sup>supported by the grants RFBR 98-01-00303 and NWO 047-008-009

<sup>2</sup>The wonderful compactification of  $\mathrm{PGL}(3, \mathbb{C})/\mathrm{PO}(3, \mathbb{C})$  was constructed by E.Study in 1886.

The Kambayashi theorem ([9], see also the exposition in [3], 5.1) implies that all normal  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ -varieties  $M$  satisfying these conditions are of the form  $[\mathrm{GL}_n]_\zeta$ .

We exploit only the constructive definition 0.1.

**0.3. Sea Urchin.** The Sea Urchin  $\mathfrak{S}\mathfrak{U}_n$  is the universal object for all the projective compactifications of  $\mathrm{GL}_n(\mathbb{C})$  in the following sense.

a) For each projective compactification  $[\mathrm{GL}_n]_\zeta$ , there exists a canonical  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ -equivariant surjective map

$$\pi_\zeta : \mathfrak{S}\mathfrak{U}_n \rightarrow [\mathrm{GL}_n]_\zeta.$$

b) Denote by  $\dot{D}_\varepsilon$  the punctured disk  $0 < |z| < \varepsilon$  in  $\mathbb{C}$ . Consider the germ in 0 of an algebraic curve

$$\dot{D}_\varepsilon \rightarrow \mathrm{GL}_n(\mathbb{C}).$$

Any such germ has a limit as  $z \rightarrow 0$  in the sea urchin  $\mathfrak{S}\mathfrak{U}_n$ . Two germs  $\gamma_1, \gamma_2$  have the same limit in the sea urchin iff for each projective compactification  $[\mathrm{GL}_n(\mathbb{C})]_\zeta$  the limits  $\lim_{z \rightarrow 0} \gamma_1(z)$  and  $\lim_{z \rightarrow 0} \gamma_2(z)$  in  $[\mathrm{GL}_n(\mathbb{C})]_\zeta$  coincide.

c)  $\pi_\zeta(\lim_{z \rightarrow 0} \gamma(z)) = \lim_{z \rightarrow 0} \pi_\zeta(\gamma(z))$  for all the germs  $\gamma$  and all the  $[\mathrm{GL}_n]_\zeta$ .

The existence of the sea urchin is obvious. Strangely enough, this object admits an explicit parametrization in elementary terms. Points of the sea urchin are enumerated by collections of integers  $(m_1, \dots, m_n)$  (defined up to a common factor) and some special collections of subspaces (hinges)  $P_1, \dots, P_\tau \subset \mathbb{C}^n \oplus \mathbb{C}^n$ .

REMARK. Obviously, there exists a universal object for all the projective compactifications in the category of compact topological spaces<sup>3</sup>. This object (for the case of Riemannian noncompact symmetric spaces) was investigated by Kushner [13], [14]. The sea urchin is not a compact space in the usual sense, hence the sea urchin differs from the Kushner compactification. The sea urchin also is simpler.  $\square$

REMARK. Obviously, the sea urchin is not a projective variety.  $\square$

**0.4. Basic observations.** Denote by  $\mathrm{Gr}_n$  the Grassmannian of  $n$ -dimensional subspaces in  $\mathbb{C}^n \oplus \mathbb{C}^n$ . Consider the canonical embedding  $\mathrm{GL}_n \rightarrow \mathrm{Gr}_n$  taking any operator to its graph.

Consider the germ in  $z = 0$  of an algebraic (or meromorphic) map  $\gamma : \dot{D}_\varepsilon \rightarrow \mathrm{GL}_n(\mathbb{C})$ . For any integer  $k$ , consider the following limit in the Grassmannian

$$R_k = \lim_{z \rightarrow 0} z^{-k} \gamma(z).$$

We obtain the bilateral sequence

$$\dots, R_{-2}, R_{-1}, R_0, R_1, R_2, \dots \quad (0.1)$$

Consider all  $k$  such that  $R_k$  is not a sum of a horizontal subspace and a vertical subspace, i.e.,

$$R_k \neq \left[ R_k \cap (\mathbb{C}^n \oplus 0) \right] \left[ R_k \cap (0 \oplus \mathbb{C}^n) \right].$$

Thus we obtain some *finite* collection of integers

$$k_1 > k_2 > \dots > k_\tau.$$

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<sup>3</sup>Let  $M_1, M_2, \dots$  be all the projective compactifications. Consider the diagonal embedding  $\mathrm{GL}_n \rightarrow M_1 \times M_2 \times \dots$  (the product is equipped with the topology of pointwise convergence = the Tihonov topology). The closure of the image of this embedding is the required universal object.

Then we select the corresponding  $R_{k_j}$  from the sequence (0.1). Thus we obtain the finite ( $\tau \leq n$ ) family of subspaces in  $\mathbb{C}^n \oplus \mathbb{C}^n$

$$\mathcal{R} = (R_{k_1}, \dots, R_{k_\tau}).$$

All possible families  $\mathcal{R}$  can easily be described, see the definition of hinges in 2.1.

Thus, for each meromorphic germ  $\gamma(z)$ , we associate the following data

$$(k_1, \dots, k_\tau; R_{k_1}, \dots, R_{k_\tau}). \quad (0.2)$$

Our paper contains two observations.

1. For any polynomial representation  $\zeta$  of  $\mathrm{GL}_n(\mathbb{C})$ , the limit of  $\zeta(\gamma(z))$  in  $[\mathrm{GL}_n]_\zeta$  is completely determined by the data (0.2)

2. The operator

$$\lim_{z \rightarrow 0} \zeta(\gamma(z))$$

admits a simple explicit description in terms of the data (0.2).

This gives the explicit description of all the spaces  $[\mathrm{GL}_n]_\zeta$  and also the description of the sea urchin.

**0.5. Structure of this paper.** Section 1 contains preliminaries on the category GA of linear relations and on the fundamental representation of GA. These objects appeared in [17], the detailed exposition is contained in [18] and [21], 2.5.

Section 2 contains the preliminaries on the hinges and the hinge semigroup. It is mainly based on [19], except Subsections 2.10, 2.11; the detailed exposition of the work [19] is present in [22]. We also explain here relations between the hinge semigroup and some well-known constructions.

Section 3 contains the construction of the sea urchin.

**0.6. Other classical groups and symmetric spaces.** In this paper, we consider only the groups  $\mathrm{GL}_n(\mathbb{C})$ . The hinge language is common for all the classical groups, all the classical symmetric varieties, and for their real forms (i.e., Riemannian and pseudo-Riemannian symmetric spaces), see [19], [22]. The results of this paper extend to this general situation more or less automatically.

**Acknowledgments.** I am grateful to C.De Concini, E.B.Vinberg and V.L.Popov for meaningful discussion of this subject. I thank the administrators of the Erwin Schrödinger Institute for Mathematical Physics, where this work was done, for their hospitality.

**Notation.** We denote by  $\mathbb{C}^*$  the multiplicative group of nonzero complex numbers.

Let  $V$  be an  $n$ -dimensional complex linear space. We denote by  $\mathrm{GL}_n(\mathbb{C})$  or  $\mathrm{GL}(V)$  the group of all invertible linear operators in  $V \simeq \mathbb{C}^n$ .

By  $\mathrm{PGL}_n(\mathbb{C})$  we denote the quotient group  $\mathrm{GL}_n(\mathbb{C})/\mathbb{C}^*$  of  $\mathrm{GL}_n(\mathbb{C})$  by the subgroup  $\mathbb{C}^*$  of all scalar matrices.

For a linear space  $Z$ , we denote by  $\mathbb{P}Z$  the projective space  $(Z \setminus 0)/\mathbb{C}^*$ .

By  $\mathrm{Mat}(Z)$  we denote the space of all linear operators in a linear space  $Z$ . By  $\mathbb{P}\mathrm{Mat}(Z)$  we denote the quotient space  $(\mathrm{Mat}(Z) \setminus 0)/\mathbb{C}^*$ .

## 1. Category GA and its fundamental representation

**1.1. Linear relations.** Let  $V, W$  be finite-dimensional linear spaces over  $\mathbb{C}$ . A *linear relation*  $P : V \rightrightarrows W$  is a linear subspace in  $V \oplus W$ .

EXAMPLE. Let  $A : V \rightarrow W$  be a linear operator. Its *graph*  $\text{graph}(A)$  consists of all vectors of the form  $v \oplus Av \in V \oplus W$ . Thus  $\text{graph}(A)$  is a linear relation  $V \rightrightarrows W$ . Below we do not distinguish linear operators and their graphs.  $\square$

Let  $V, W, Y$  be linear spaces, and let  $P : V \rightrightarrows W$ ,  $Q : W \rightrightarrows Y$  be linear relations. Their product  $S = QP$  is the linear relation  $S : V \rightrightarrows Y$  consisting of all  $v \oplus y \in V \oplus Y$  such that there exists  $w \in W$  satisfying the conditions

$$v \oplus w \in P, \quad w \oplus y \in Q.$$

For any linear relation  $P : V \rightrightarrows W$ , we define

- a) the *kernel*  $\text{Ker } P \subset V$  is  $P \cap (V \oplus 0)$ ;
- b) the *image*  $\text{Im } P \subset W$  is the image of the projection of  $P \subset V \oplus W$  on  $W$  along  $V$ ,
- c) the *domain*  $\text{Dom } P \subset V$  is the image of the projection of  $P \subset V \oplus W$  on  $V$  along  $W$ ,
- d) the *indefiniteness*  $\text{Indef } P \subset W$  is  $P \cap (0 \oplus W)$ ,
- e) the *dimension*  $\dim P$  is the dimension of  $P$ ,
- f) the *rank*

$$\begin{aligned} \text{rk } P &:= \dim P - \dim \text{Ker } P - \dim \text{Indef } P = \\ &= \dim \text{Dom } P - \dim \text{Ker } P = \dim \text{Im } P - \dim \text{Indef } P. \end{aligned}$$

REMARK. Obviously, for any linear operators  $A : V \rightarrow W$ ,  $B : W \rightarrow Y$ ,

$$\begin{aligned} \text{graph}(BA) &= \text{graph}(B) \text{ graph}(A); \\ \text{Ker } \text{graph}(A) &= \text{Ker } A; \quad \text{Im } \text{graph}(A) = \text{Im } A \\ \text{rk } \text{graph}(A) &= \text{rk } A. \end{aligned}$$

For  $P : V \rightrightarrows W$  we define the *pseudoinverse* linear relation  $P^\square : W \rightrightarrows V$ . It is the same subspace  $P \subset V \oplus W$  regarded as a subspace in  $W \oplus V$ .

For  $P : V \rightrightarrows W$  and  $c \in \mathbb{C}^*$  we define the linear relation  $c \cdot P$  consisting of all vectors  $v \oplus cw$ , where  $v \oplus w$  ranges in  $P$ .

REMARK. Let  $A$  be a linear operator. Then  $c \cdot \text{graph}(A) = \text{graph}(cA)$ . For an invertible linear operator  $A : V \rightarrow V$ , we have

$$(\text{graph } A)^\square = \text{graph}(A^{-1}).$$

**1.2. Category GA.** The objects of the category GA are finite-dimensional linear spaces over  $\mathbb{C}$ . Set of morphisms  $\text{Mor}(V, W) = \text{Mor}_{\text{GA}}(V, W)$  consists of all linear relations  $P : V \rightrightarrows W$  and the formal element  $\text{null}_{V, W}$ .

REMARK. The dimension of  $P$  is an arbitrary number  $0, 1, 2, \dots, \dim V + \dim W$ . The element  $\text{null}_{V, W}$  can not be identified with a linear relation.  $\square$

Let  $P : V \rightrightarrows W$ ,  $Q : W \rightrightarrows Y$  be linear relations. If

$$\text{Im } P + \text{Dom } Q = W; \tag{1.1}$$

$$\text{Indef } P \cap \text{Ker } Q = 0, \tag{1.2}$$

then the product  $QP$  in the category GA coincides with the product of linear relations. Otherwise,

$$QP = \text{null}_{V, Y}.$$

The product of null and any morphism is null.

PROPOSITION 1.1. (see [21], 2.7) a) *For any linear spaces  $V, W, Y, Z$  and any morphisms  $P \in \text{Mor}(V, W), Q \in \text{Mor}(W, Y), R \in \text{Mor}(Y, Z)$ , the associativity holds*

$$(RQ)P = R(QP).$$

b) *Let  $P \in \text{Mor}(V, W), Q \in \text{Mor}(W, Y)$ . If  $QP \neq \text{null}$ , then*

$$\dim(QP) = \dim Q + \dim P - \dim W.$$

REMARK. The group  $\text{Aut}(V)$  of automorphisms of an object  $V$  is the group  $\text{GL}(V)$ .  $\square$

**1.3. Semigroup  $\Gamma(V)$ .** Denote by  $\Gamma(V)$  the subset in  $\text{Mor}(V, V)$  consisting of  $\text{null}_{V,V}$  and all linear relations  $R : V \rightrightarrows V$  such that  $\dim R = \dim V$ .

By Proposition 1.1,  $\Gamma(V)$  is closed with respect to the multiplication.

Also, for a linear relation  $R \in \Gamma(V)$

$$\dim \text{Dom } R + \dim \text{Indef } R = \dim V; \quad \dim \text{Ker } R + \dim \text{Im } R = \dim V$$

**1.4. Exterior algebras.** Let  $V$  be a complex linear space. Denote by  $\Lambda(V)$  the exterior algebra of the space  $V$ . Recall that  $\Lambda(V)$  is the associative algebra with generators  $v$ , where  $v$  ranges in  $V$ , and the relations

$$\begin{aligned} v \wedge w &= -w \wedge v, \quad ; \\ (\mu v_1 + \nu v_2) \wedge w &= \mu(v_1 \wedge w) + \nu(v_2 \wedge w), \end{aligned}$$

where  $\mu, \nu \in \mathbb{C}, v, w \in V$  and the sign  $\wedge$  denotes the multiplication in  $\Lambda(V)$ .

We denote by  $\Lambda^k V$  the linear subspace in  $\Lambda(V)$  spanned by all vectors having the form

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k, \quad v_j \in V.$$

The space  $\Lambda^k V$  is called *the  $k$ -th exterior power of  $V$* . If  $e_1, \dots, e_n$  is a basis in  $V$ , then the collection  $e_{i_1} \wedge \cdots \wedge e_{i_k}$ , where  $i_1 < i_2 < \cdots < i_k$ , is a basis in  $\Lambda^k V$ .

Let  $A : V \rightarrow W$  be a linear operator. We define the linear *operator of change of the variables*

$$\lambda_{\text{cha}}(A) : \Lambda V \rightarrow \Lambda W$$

by

$$\lambda_{\text{cha}}(A)v_1 \wedge \cdots \wedge v_k = Av_1 \wedge \cdots \wedge Av_k.$$

If  $A$  is an operator  $V \rightarrow W$ , and  $B$  is an operator  $W \rightarrow Y$ , then

$$\lambda_{\text{cha}}(B)\lambda_{\text{cha}}(A) = \lambda_{\text{cha}}(BA).$$

The operators  $\lambda_{\text{cha}}(A)$  preserve the degree  $k$ , and hence we also obtain the operators in the exterior powers

$$\lambda_{\text{cha}}^k(A) : \Lambda^k V \rightarrow \Lambda^k W.$$

**1.5. Fundamental representation of the category GA.** Let  $S : V \rightrightarrows W$  be a linear relation. Then there exist a basis

$$f_1, \dots, f_\alpha, g_1, \dots, g_\beta, h_1, \dots, h_\gamma \tag{1.3}$$

in  $V$  and a basis

$$F_1, \dots, F_\mu, G_1, \dots, G_\beta, H_1, \dots, H_\nu \quad (1.4)$$

in  $W$  such that  $S \subset V \oplus W$  is spanned by the vectors

$$0 \oplus F_1, \dots, 0 \oplus F_\mu, g_1 \oplus G_1, \dots, g_\beta \oplus G_\beta, h_1 \oplus 0, \dots, h_\gamma \oplus 0. \quad (1.5)$$

REMARK. Thus,

- the vectors  $h_i$  form a basis in  $\text{Ker } S$ ;
- the vectors  $h_i$  and  $g_j$  form a basis in  $\text{Dom } S$ ;
- the vectors  $F_k$  form a basis in  $\text{Indef } S$ ;
- the vectors  $F_k, G_j$  form a basis in  $\text{Im } S$ .

□

REMARK. Let  $V = W$  and  $S \in \Gamma(V)$ . Then  $\alpha = \mu, \gamma = \nu$ .

□

We define the linear operator

$$\lambda(S) : \Lambda(V) \rightarrow \Lambda(W)$$

by

$$\lambda(S)f_1 \wedge f_2 \wedge \dots \wedge f_\alpha \wedge g_{i_1} \wedge \dots \wedge g_{i_k} = F_1 \wedge F_2 \wedge \dots \wedge F_\mu \wedge G_{i_1} \wedge \dots \wedge G_{i_k},$$

and

$$\lambda(S)\xi = 0$$

for all other basic vectors  $\xi$  in  $\Lambda V$ .

REMARK. The bases (1.3), (1.4) are not uniquely determined by the linear relation  $S$ . If we change the bases, then the operator  $\lambda(S)$  shall be multiplied by a nonzero constant. □

Let  $A \in \text{GL}(W)$ ,  $B \in \text{GL}(V)$ . Then

$$\lambda(ASB) = \lambda_{\text{cha}}(A)\lambda(S)\lambda_{\text{cha}}(B).$$

We also assume

$$\lambda(\text{null}) = 0.$$

REMARK. Let  $S$  be a graph of a linear operator  $A$ . Then  $\lambda(S) = \lambda_{\text{cha}}(A)$ . Nevertheless,  $\lambda_{\text{cha}}(A)$  is a well defined operator in  $\Lambda V$ , the operator  $\lambda(S)$  is defined up to a nonzero scalar factor. By this reason, we preserve the both notations  $\lambda(\cdot)$  and  $\lambda_{\text{cha}}(\cdot)$ , since their meanings slightly differ. □

THEOREM 1.3. (see [21], II.7 or [22], §1) a) *Let  $P : V \rightrightarrows W$ ,  $Q : W \rightrightarrows Y$  be linear relations. Then*

$$\lambda(Q)\lambda(P) = c(Q, P)\lambda(QP),$$

where  $c(q, p) \in \mathbb{C}$ .

b)  $c(q, p) \neq 0$  iff  $QP \neq \text{null}$ .

REMARK. For the coordinateless definition of the operators  $\lambda(P)$ , see [18], [21]

REMARK. By the construction, the operator  $\lambda(P)$  takes homogeneous vectors to homogeneous vectors. Thus we obtain the family of the operators

$$\lambda^k(P) : \Lambda^k V \rightarrow \Lambda^{k+\dim P - \dim V} W.$$

**1.6. Fundamental representations of the semigroup  $\Gamma(V)$ .** The main tool below is the semigroup  $\Gamma(V)$  defined in 1.3. Obviously, for  $P \in \Gamma(V)$  we have

$$\lambda^m(P) : \Lambda^m V \rightarrow \Lambda^m V.$$

Thus we obtain the collection of the projective representations  $\lambda^m$  of the semigroup  $\Gamma(V)$  in the spaces  $\Lambda^m V$ .

LEMMA 1.3. Consider  $P \in \Gamma(V)$ .

- a)  $\lambda^m(P) \neq 0$  iff  $\dim \text{Indef } P \leq m \leq \dim \text{Im } P$
- b) If  $m = \dim \text{Indef } P$ , then  $\text{rk } \lambda^m(P) = 1$ , and

$$\lambda^m(P) = \lambda^m(Q)$$

for any  $Q \in \Gamma(V)$  such that  $\text{Indef } Q = \text{Indef } P$ ,  $\text{Dom } Q = \text{Dom } P$ . In particular, we can choose  $Q = \text{Dom } P \oplus \text{Indef } P$ .

- c) If  $m = \dim \text{Im } P$ , then  $\text{rk } \lambda^m(P) = 1$ , and

$$\lambda^m(P) = \lambda^m(R)$$

for any  $R \in \Gamma(V)$  such that  $\text{Im } R = \text{Im } P$ ,  $\text{Ker } R = \text{Ker } P$ . In particular, we can choose  $R = \text{Ker } P \oplus \text{Im } P$ .

PROOF. Consider the canonical form described in 1.5. The conditions  $\dim V = \dim W = \dim P$  imply  $\alpha = \mu$ ,  $\gamma = \nu$ . Now all statements become obvious, see also [22].  $\square$

PROPOSITION 1.4. The map  $\lambda(P)$  is a continuous map from the Grassmannian  $\text{Gr}_n$  of all  $n$ -dimensional subspaces in  $V \oplus V$  to  $\mathbb{P}\text{Mat}(\Lambda V)$ . In particular, its image is a closed subset in  $\mathbb{P}\text{Mat}(\Lambda V)$ .

## 2. Hinges

This Section contains the preliminaries on the hinges with short explanations and sketches of proofs. For more details, see [22].

In this Section,  $V$  is an  $n$ -dimensional complex linear space and  $\text{Gr}_n$  is the Grassmannian of all  $n$ -dimensional linear subspaces in  $V \oplus V \simeq \mathbb{C}^n \oplus \mathbb{C}^n$ .

**2.1. Hinges.** A hinge

$$\mathcal{P} = (P_1, \dots, P_k) : V \rightrightarrows V$$

is a family of linear relations  $P_j : V \rightrightarrows V$  such that  $\dim P_j = \dim V = n$  (hence  $P_j \in \Gamma(V)$ ) and

$$\text{Ker } P_j = \text{Dom } P_{j+1}, \quad \text{where } j = 1, 2, \dots, k-1, \quad (2.1)$$

$$\text{Im } P_j = \text{Indef } P_{j+1}, \quad \text{where } j = 1, 2, \dots, k-1, \quad (2.2)$$

$$\text{Dom } P_1 = V, \quad (2.3)$$

$$\text{Im } P_k = V, \quad (2.4)$$

$$P_j \neq \text{Ker } P_j \oplus \text{Indef } P_j, \quad \text{where } j = 1, 2, \dots, k. \quad (2.5)$$

### 2.2. Comments on the definition.

REMARK. The conditions (2.3)–(2.4) are the interpretation of the conditions (2.1)–(2.2) for  $j = 0$  and  $j = k$ . The condition (2.3) means that the first term  $P_1$  of a hinge is an operator. The condition (2.4) means that the last term  $P_k$  is a linear relation pseudoinverse to an operator.  $\square$

REMARK. The graph of an *invertible* operator is a hinge ( $k = 1$ ). The graph of a noninvertible operator is not a hinge.  $\square$

REMARK. Let  $A : V \rightarrow V$ ,  $B : V \rightarrow V$  be linear operators such that

$$\text{Im } A = \text{Ker } B; \quad \text{Ker } A = \text{Im } B.$$

Then

$$(\text{graph}(A), \text{graph}(B)^\square)$$

is a hinge. Any hinge consisting of two terms ( $k = 2$ ) has this form.  $\square$

REMARK. The condition (2.5) is equivalent to the condition

$$\text{rk } P_j > 0.$$

REMARK. The condition (2.5) is technical. For each hinge  $\mathcal{P}$  we define the *completed hinge*

$$\widehat{\mathcal{P}} := (Q_0, P_1, Q_1, P_2, Q_2, \dots, P_k, Q_k), \quad (2.6)$$

where

$$\begin{aligned} Q_0 &= V \oplus 0, \\ Q_j &= \text{Ker } P_j \oplus \text{Im } P_j = \text{Dom } P_{j+1} \oplus \text{Indef } P_{j+1}, \\ Q_k &= 0 \oplus V, \end{aligned} \quad (2.7)$$

Obviously,  $\mathcal{P}$  is uniquely determined by  $\widehat{\mathcal{P}}$ . Thus, the space of all hinges and the space of all completed hinges coincide.  $\square$

REMARK. We have

$$\begin{aligned} V &\supset \text{Ker } P_1 \supset \text{Ker } P_2 \supset \dots \supset \text{Ker } P_k = 0, \\ 0 &\subset \text{Im } P_1 \subset \text{Im } P_2 \subset \dots \subset \text{Im } P_k = V. \end{aligned}$$

By (2.5),  $\text{Ker } P_{j+1} \neq \text{Ker } P_j$ . This implies  $k \leq n$ .  $\square$

REMARK. We have

$$\text{Ker } P_j \oplus \text{Indef } P_j \subset P_j \subset \text{Dom } P_j \oplus \text{Im } P_j.$$

The image of  $P_j$  under the natural projection

$$\text{Dom } P_j \oplus \text{Im } P_j \longrightarrow (\text{Dom } P_j / \text{Ker } P_j) \oplus (\text{Im } P_j / \text{Indef } P_j)$$

is a graph of an invertible operator

$$\text{Dom } P_j / \text{Ker } P_j \rightarrow \text{Im } P_j / \text{Indef } P_j.$$

REMARK. Thus, hinges can be defined in the following way. Consider two flags

$$\begin{aligned} 0 &= Y_0 \subset Y_1 \subset \dots \subset Y_k = V, \\ V &= Z_0 \supset Z_1 \supset \dots \supset Z_k = 0, \end{aligned}$$

such that

$$\dim Y_j / Y_{j-1} = \dim Z_{j-1} / Z_j \quad \text{for all } j.$$

For each  $j$ , we fix an invertible linear operator

$$A_j : Y_j / Y_{j-1} \rightarrow Z_{j-1} / Z_j.$$

By the previous remark, these data define a hinge.  $\square$



**2.3. Notation.** We denote by  $\text{Hinge}(V) = \text{Hinge}_n$  the space of all hinges  $\mathcal{P} : V \rightrightarrows V$ .

We denote by  $\text{Hinge}^*(V) = \text{Hinge}_n^*$  the space of all hinges  $\mathcal{P} = (P_1, \dots, P_k) : V \rightrightarrows V$  defined up to the equivalence

$$(P_1, \dots, P_k) \sim (c_1 P_1, \dots, c_k P_k), \quad \text{where } c_j \in \mathbb{C}^*.$$

Considering 1-term hinges ( $k = 1$ ), we obtain

$$\text{Hinge}_n \supset \text{GL}_n(\mathbb{C}); \quad \text{Hinge}_n^* \supset \text{PGL}_n(\mathbb{C}).$$

**2.4. Topology on  $\text{Hinge}_n^*$ .** Below we define a structure of an irreducible smooth projective algebraic variety on  $\text{Hinge}_n^*$ . In this Subsection, we define the topology on  $\text{Hinge}_n^*$ .

**2.4.a. Convergence of sequences  $g_j \in \text{PGL}_n$  to points of  $\text{Hinge}_n^*$ .** The sequence  $g_j \in \text{PGL}_n$  converges to  $\mathcal{P} = (P_1, \dots, P_k) \in \text{Hinge}_n^*$ , if there exist  $k$  sequences

$$\beta_1^{(1)}, \beta_2^{(1)}, \beta_3^{(1)}, \dots; \quad \beta_1^{(2)}, \beta_2^{(2)}, \beta_3^{(2)}, \dots; \quad \dots; \quad \beta_1^{(k)}, \beta_2^{(k)}, \beta_3^{(k)}, \dots, \quad (2.8)$$

( $\beta_j^{(\sigma)} \in \mathbb{C}^*$ ) such that  $\beta_j^{(\sigma)} g_j$  converge to  $P_\sigma$  in  $\text{Gr}_n$  for all  $\sigma = 1, \dots, k$ .

REMARK. This implies

$$\lim_{j \rightarrow \infty} \beta_j^{(\sigma)} / \beta_j^{(\tau)} = \infty \quad \text{for } \sigma > \tau.$$

EXAMPLE. Let

$$g_j = \begin{pmatrix} 4^j & 0 & 0 \\ 0 & 2^j & 0 \\ 0 & 0 & 2^{-j} \end{pmatrix}.$$

Then we can choose

$$\beta_j^{(1)} = 4^{-j}; \quad \beta_j^{(2)} = 2^{-j}; \quad \beta_j^{(3)} = 2^j.$$

For the sequences

$$\mu_j^{(0)} = 8^{-j}; \quad \mu_j^{(1)} = 2^{-3/2 j}; \quad \mu_j^{(2)} = 1; \quad \mu_j^{(3)} = 4^j$$

the limits  $Q_j = \lim_{j \rightarrow \infty} \mu_j^{(\sigma)} g_j$  in  $\text{Gr}_n$  also exist, but all the limits  $Q_0, \dots, Q_3$  have rank 0. These limits are the elements  $Q_j$  of the completed hinge, see (2.6).  $\square$

LEMMA 2.1. *Any sequence  $g_j \in \text{PGL}_n(\mathbb{C})$  contains a subsequence convergent in our sense.*

PROOF. We represent  $g_j$  in the form

$$g_j = A_j \begin{pmatrix} u_j^{(1)} & 0 & \dots \\ 0 & u_j^{(2)} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} B_j,$$

where  $A_j, B_j$  are unitary matrices, and

$$u_j^{(1)} \geq u_j^{(2)} \geq \dots > 0$$

Selecting a subsequence, we can assume that

1) the sequences  $A_j, B_j$  are convergent

$$2) \exists \lim_{j \rightarrow \infty} u_j^{(m)} / u_j^{(1)} = \alpha_m \text{ for all } m > 1.$$

Obviously,  $1 \geq \alpha_2 \geq \alpha_3 \geq \dots \geq 0$ . Consider  $\tau$  such that  $\alpha_{\tau-1} \neq 0$ ,  $\alpha_\tau = 0$ . After the next selection of a subsequence, we can assume

$$3) \exists \lim_{j \rightarrow \infty} u_j^{(m)} / u_j^{(\tau)} = \beta_m \text{ for all } m > \tau. \text{ Obviously, } 1 \geq \beta_{\tau+1} \geq \beta_{\tau+2} \geq \dots \geq 0.$$

Then we repeat the same argument again, again, and again.

Now we assume  $\beta_j^{(1)} = (u_j^{(1)})^{-1}$ ,  $\beta_j^{(2)} = (u_j^{(\tau)})^{-1}$ , etc.  $\square$

Thus the space  $\text{Hinge}_n^* \setminus \text{PGL}_n(\mathbb{C})$  is some kind of boundary of  $\text{PGL}_n(\mathbb{C})$ .

**2.4.b. Convergence on the boundary.** A sequence  $\mathcal{P}^j = (P_1^j, \dots, P_k^j)$  converges to a hinge  $\mathcal{Q} = (Q_1, \dots, Q_l)$ , if for any  $Q_u$ , there exist  $v = 1, \dots, k$  and a sequence  $\mu_j \in \mathbb{C}^*$  such that  $\mu_j P_v^j$  converges to  $Q_u$  in  $\text{Gr}_n$ .

**2.4.c. Formal description of the structure of a compact metric space on  $\text{Hinge}_n^*$ .** Consider the action of the group  $\mathbb{C}^*$  on  $\text{Gr}_n$  given by  $P \mapsto c \cdot P$ . Fixed points of  $\mathbb{C}^*$  in  $\text{Gr}_n$  are linear relations of rank 0. In other words, the fixed points have the form

$$Q = \text{Ker } Q \oplus \text{Indef } Q.$$

All other orbits of  $\mathbb{C}^*$  in  $\text{Gr}_n$  have trivial stabilizers. If  $P \in \text{Gr}_n$  is not a fixed point of  $\mathbb{C}^*$ -action, then the closure of the orbit  $\mathbb{C}^* \cdot P$  consists of the orbit itself and of the pair of the points

$$\text{Dom } P \oplus \text{Indef } P, \quad \text{Ker } P \oplus \text{Im } P.$$

For a hinge  $\mathcal{P} = (P_1, \dots, P_k)$ , we define the subset  $\Omega(\mathcal{P})$  in  $\text{Gr}_n$  by

$$\Omega(\mathcal{P}) := Q_0 \cup \mathbb{C}^* P_1 \cup Q_1 \cup \mathbb{C}^* P_2 \cup Q_2 \cup \dots \cup Q_{k-1} \cup \mathbb{C}^* P_k \cup Q_k,$$

where the points  $Q_j = \text{Ker } P_j \oplus \text{Im } P_j = \text{Dom } P_{j+1} \oplus \text{Indef } P_{j+1}$  are the elements of the completed hinge, see (2.6).

We emphasize that the closure of  $\mathbb{C}^* P_j$  contains  $Q_{j-1}$  and  $Q_j$ . Hence the subset  $\Omega(\mathcal{P})$  is closed and connected.

Denote by  $\text{Close}(\text{Gr}_n)$  the space of all closed subsets of the Grassmannian  $\text{Gr}_n$ . Consider an arbitrary metric  $\rho$  on  $\text{Gr}_n$  compatible with the topology. For  $x \in \text{Gr}_n$  and  $A \in \text{Close}(\text{Gr}_n)$ , we define the distance

$$\rho(x, A) = \min_{y \in A} \rho(x, y).$$

The *Hausdorff metric* in  $\text{Close}(\text{Gr}_n)$  is defined by

$$d(A, B) = \max \left[ \max_{x \in A} d(x, B), \max_{y \in B} d(y, A) \right].$$

**THEOREM 2.2 a)** *The image of the embedding  $\text{Hinge}_n^* \rightarrow \text{Close}(\text{Gr}_n)$  given by  $\mathcal{P} \mapsto \Omega(\mathcal{P})$  is a compact subset in  $\text{Close}(\text{Gr}_n)$ . Thus we obtain a topology of a compact metric space on  $\text{Hinge}_n^*$ .*

**b)** *The group  $\text{PGL}_n(\mathbb{C})$  is dense in  $\text{Hinge}_n^*$ .*

**REMARK.** The space of orbits of  $\mathbb{C}^*$  on  $\text{Gr}_n$  is a nonseparated topological space. Construction described above is the result of an application of the construction of the Hausdorff quotient described in [20],[22]. In the algebraic geometry, there exist also more delicate constructions of the Hilbert scheme quotient and the Chow scheme quotient, see [1],[10]; in our case, these constructions are equivalent to the Hausdorff quotient.  $\square$

**2.5. Orbits of the group  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$  on  $\mathrm{Hinge}_n$ .** The group  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$  acts on  $V \oplus V \simeq \mathbb{C}^n \oplus \mathbb{C}^n$  in the obvious way. Hence it acts on the spaces  $\mathrm{Hinge}_n$  and  $\mathrm{Hinge}_n^*$ .

LEMMA 2.3. *The group  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$  has  $2^n$  orbits on the space  $\mathrm{Hinge}_n$ . These orbits are enumerated by the number  $k = 1, 2, \dots, n$  and the positive numbers*

$$\alpha_1 = \mathrm{rk} P_1, \dots, \alpha_k = \mathrm{rk} P_k, \quad \text{where } \alpha_1 + \dots + \alpha_k = n.$$

PROOF. Obvious. □

We denote these orbits by

$$\mathcal{O}[\alpha] = \mathcal{O}[\alpha_1, \dots, \alpha_k].$$

Fix a basis  $e_1, \dots, e_n \in V$ . For a given collection  $\alpha_1, \dots, \alpha_k$ , we define the *canonical hinge*  $\mathcal{P}_{\alpha_1, \dots, \alpha_k} \in \mathcal{O}[\alpha_1, \dots, \alpha_k]$  by

$$\mathcal{P}_{\alpha_1, \dots, \alpha_k} = (P_1, \dots, P_k) : V \rightrightarrows V,$$

where the linear relation  $P_j \subset V \oplus V$  is spanned by the vectors

$$\begin{aligned} 0 \oplus e_\sigma, & \quad \text{where } \sigma \leq \alpha_1 + \dots + \alpha_{j-1}, \\ e_\tau \oplus e_\tau, & \quad \text{where } \alpha_1 + \dots + \alpha_{j-1} < \tau \leq \alpha_1 + \dots + \alpha_j, \\ e_\mu \oplus 0, & \quad \text{where } \mu > \alpha_1 + \dots + \alpha_j. \end{aligned}$$

REMARK. Denote by  $\mathcal{O}^*[\alpha]$  the image of  $\mathcal{O}[\alpha]$  in  $\mathrm{Hinge}_n^*$ . We have

$$\dim \mathcal{O}[\alpha] = n^2; \quad \dim \mathcal{O}^*[\alpha_1, \dots, \alpha_\tau] = n^2 - \tau$$

## 2.6. Alternative.

THEOREM 2.4. *Let  $\mathcal{P} = (P_1, \dots, P_k) : V \rightrightarrows V$  be a hinge. Fix  $m = 0, 1, \dots, n$ . Consider the family of operators*

$$\lambda^m(P_1), \lambda^m(P_2), \dots, \lambda^m(P_k) : \Lambda^m V \rightarrow \Lambda^m V. \quad (2.9)$$

*Then there are only two possibilities.*

- 1) *There exists a unique  $j$  such that  $\lambda^m(P_j) \neq 0$ .*
- 2) *There exists  $j$  such that  $\lambda^m(P_j) \neq 0$ ,  $\lambda^m(P_{j+1}) \neq 0$ , and  $\lambda^m(P_\tau) = 0$  for all  $\tau \neq j, j+1$ . In this case,  $\lambda^m(P_j)$  and  $\lambda^m(P_{j+1})$  have rank 1 and coincide up to a nonzero factor. They also coincide with  $\lambda(Q_j)$ , where  $Q_j = \mathrm{Ker} P_j \oplus \mathrm{Im} P_j$  is the term of the completed hinge  $\widehat{\mathcal{P}}$ , see (2.6).*

PROOF. This is a consequence of Lemma 1.4. □

Now for each hinge  $\mathcal{P} = (P_1, \dots, P_k) : V \rightrightarrows V$  and for each  $m = 0, 1, \dots, n$ , we define the operator

$$\lambda^m(\mathcal{P}) : \Lambda^m V \rightarrow \Lambda^m V$$

as the unique nonzero term of the sequence (2.9). By the construction, this operator is defined up to a nonzero factor.

REMARK. Obviously, for any  $g_1, g_2 \in \mathrm{GL}_n(\mathbb{C})$ ,

$$\lambda^m(g_1 \mathcal{P} g_2) = \lambda_{\mathrm{cha}}^m(g_1) \lambda^m(\mathcal{P}) \lambda_{\mathrm{cha}}^m(g_2).$$

**2.7. Example: the operators  $\lambda^m(\mathcal{P})$  for canonical hinges.** Consider the canonical hinge  $\mathcal{P}_\alpha = \mathcal{P}_{\alpha_1, \dots, \alpha_k}$ . We intend to describe the operator  $\mathcal{L}^m := \lambda^m(\mathcal{P}_\alpha)$ . Assume

$$u_j = \alpha_1 + \dots + \alpha_{j-1}$$

Let  $u_j \leq m \leq u_{j+1}$ , let  $s := m - u_j$  and  $u_j < l_1 < \dots < l_s \leq u_{j+1}$ . Then for each element

$$h = e_1 \wedge e_2 \wedge e_3 \wedge \dots \wedge e_{u_j} \wedge e_{l_1} \wedge \dots \wedge e_{l_s}$$

of the standard basis, we have

$$\lambda^m(\mathcal{P}_\alpha)h = h;$$

and  $\lambda^m(\mathcal{P}_\alpha)$  annihilates all other elements  $e_{t_1} \wedge \dots \wedge e_{t_m}$  of the standard basis in  $\Lambda^m V$ .

**2.8. The projective embedding of  $\text{Hinge}_n^*$ .** In 2.6, for any  $\mathcal{P} \in \text{Hinge}_n$ , we constructed the family of nonzero linear operators

$$\lambda^\circ(\mathcal{P}) := (\lambda^1(\mathcal{P}), \lambda^2(\mathcal{P}), \dots, \lambda^{n-1}(\mathcal{P})) \quad (2.10)$$

defined up to nonzero factors. Consider two hinges

$$\mathcal{P} = (P_1, \dots, P_k); \quad \mathcal{R} = (c_1 P_1, \dots, c_k P_k), \quad \text{where } c_j \in \mathbb{C}^*.$$

Obviously, the operators  $\lambda^m(\mathcal{P})$  and  $\lambda^m(\mathcal{R})$  coincide up to a nonzero factor.

Thus we obtain the map

$$\lambda^\circ : \text{Hinge}_n^* \rightarrow \bigtimes_{m=1}^{n-1} \mathbb{P}\text{Mat}(\Lambda^m V).$$

Consider also the map

$$\lambda^\circ : \text{PGL}_n(\mathbb{C}) \rightarrow \bigtimes_{m=1}^{n-1} \mathbb{P}\text{Mat}(\Lambda^m V)$$

given by

$$\lambda^\circ(g) := (\lambda_{\text{cha}}^1(g), \dots, \lambda_{\text{cha}}^{n-1}(g)). \quad (2.11)$$

**THEOREM 2.5.** ([19],[22]) *The map (2.10) from  $\text{Hinge}_n^*$  to  $\bigtimes_{m=1}^{n-1} \mathbb{P}\text{Mat}(\Lambda^m V)$  is continuous.*

**COROLLARY 2.6.** a) *The image of the map (2.10) is compact.*

b) *The image of the map (2.10) is the closure of the  $\lambda^\circ(\text{PGL}_n(\mathbb{C}))$ .*

c) *The space  $\text{Hinge}_n^*$  is an irreducible projective variety.*

**SKETCH OF PROOF OF THEOREM 2.5.** We shall prove the implication

$$g_j \in \text{GL}_n(\mathbb{C}) \text{ converges to } \mathcal{P} \implies \lambda^\circ(g_j) \text{ converges to } \lambda^\circ(\mathcal{P}). \quad (2.12)$$

Let us represent  $g_j$  in the form

$$g_j = A_j D_j B_j,$$

where  $A_j, B_j$  are unitary matrices and  $D_j$  are diagonal matrices with the decreasing eigenvalues. We can assume that the sequences  $A_1, A_2, \dots$  and  $B_1, B_2, \dots$  are convergent. Thus the question is reduced to the case  $A_j = 1, B_j = 1$ . For this case, the statement can easily be checked.

Corollary 2.6 implies the following consequence.

**COROLLARY 2.7.** a) *The space  $\text{Hinge}_n^*$  is an irreducible smooth projective variety.*

b) *The variety  $\text{Hinge}_n^*$  coincides with the Semple complete collineation space.*

c) The variety  $\text{Hinge}_n^*$  coincides with the De Concini–Procesi [4] compactification of  $\text{PGL}_n(\mathbb{C})$ .

By the definition (see [28]), the Semple complete collineation space is the closure of the image of the map (2.11). By the Semple theorem, the complete collineation space is a smooth projective variety, and by [4] it coincides with the De Concini–Procesi compactification, see also below 2.13.

**2.9. Semigroup of hinges.** Let us define one more variation of the space  $\text{Hinge}_n$ . Denote by  $\widetilde{\text{Hinge}}_n$  the set of all elements of  $\bigtimes_{m=0}^n \text{Mat}(\Lambda^m V)$  having the form

$$\mathcal{A} = (A_0, \dots, A_n) = (c_0 \cdot \lambda^0(\mathcal{P}), c_1 \cdot \lambda^1(\mathcal{P}), c_2 \cdot \lambda^2(\mathcal{P}), \dots, c_n \cdot \lambda^n(\mathcal{P})),$$

where  $\mathcal{P}$  is a hinge and  $c_0, \dots, c_n \in \mathbb{C}$ . We say that  $\mathcal{A}$  lies over  $\mathcal{P}$ . If all  $c_j$  are nonzero, we say that  $\mathcal{A}$  is *nondegenerated*.

PROPOSITION 2.8.  $\widetilde{\text{Hinge}}_n$  is a subsemigroup in  $\bigtimes_{m=0}^n \text{Mat}(\Lambda^m V)$ .

This easily follows from Theorem 2.5.

Now we intend to give a constructive description of the product in  $\widetilde{\text{Hinge}}_n$ .

Let  $\mathcal{R} = (R_1, \dots, R_s)$  be a family of linear relations  $V \rightrightarrows V$ , and  $\dim R_j = n$ . We say that  $\mathcal{R}$  is a *weak hinge* if for each  $j$

$$\text{Ker } R_j \supset \text{Dom } R_{j+1},$$

$$\text{Im } R_j \subset \text{Indef } R_{j+1}.$$

REMARK. Let  $\widehat{\mathcal{P}} = (Q_0, P_1, Q_1, P_2, \dots, P_k, Q_k)$  be a completed hinge (see (2.6)). Then any subcollection of  $\widehat{\mathcal{P}}$  is a weak hinge, and each weak hinge can be obtained in this way.  $\square$

For any weak hinge  $\mathcal{R} = (R_1, \dots, R_s)$  and any  $m = 0, 1, \dots, n$ , we intend to construct the canonical operator

$$\lambda^m(\mathcal{R}) : \Lambda^m V \rightarrow \Lambda^m V$$

defined up to a scalar factor. For this, we consider the sequence

$$\lambda^m(R_1), \lambda^m(R_2), \dots, \lambda^m(R_s).$$

If this family contains a nonzero term  $\lambda^m(R_j)$ , then

$$\lambda^m(\mathcal{R}) := \lambda^m(R_j).$$

Otherwise,

$$\lambda^m(\mathcal{R}) := 0.$$

THEOREM 2.9. ([19], [22]) Let  $\mathcal{R} = (R_1, \dots, R_s)$ ,  $\mathcal{T} = (T_1, \dots, T_\tau)$  be weak hinges. Then the family of all

$$T_i R_j \neq \text{null}$$

is a weak hinge.

SKETCH OF PROOF. Let  $T_i R_j \neq \text{null}$ . It can easily be checked that the segments

$$\dim \text{Indef } R_j \leq m \leq \dim \text{Im } R_j, \tag{2.13}$$

$$\dim \text{Indef } T_i \leq m \leq \dim \text{Im } T_i \tag{2.14}$$

have nonzero intersection. After this remark, Theorem 2.8 can easily be checked.  $\square$

THEOREM 2.10. ([19], [22]) *For each  $m$ ,*

$$\lambda^m(\mathcal{T})\lambda^m(\mathcal{R}) = c \cdot \lambda^m(\mathcal{TR}),$$

where  $c \in \mathbb{C}^*$ .

PROOF. Assume  $m$  satisfies the equations (2.13)–(2.14). Then

$$\lambda^m(\mathcal{T}) = \lambda^m(T_i); \quad \lambda^m(\mathcal{R}) = \lambda^m(R_j).$$

Thus,

$$\lambda^m(\mathcal{T})\lambda^m(\mathcal{R}) = \lambda^m(T_i)\lambda^m(R_j) = \lambda^m(T_i R_j) = \lambda^m(\mathcal{TR}). \quad \square$$

Thus, for  $\mathcal{A}$  lying over a weak hinge  $\mathcal{P}$  and  $\mathcal{B}$  lying over a weak hinge  $\mathcal{Q}$ , the product  $\mathcal{AB}$  lies over  $\mathcal{PQ}$ .

**2.10. Canonical embedding**  $\text{Hinge}_n \rightarrow \widetilde{\text{Hinge}_n}$ . Let  $\mathcal{P} \in \text{Hinge}_n$ . The construction 2.6 defines the operators  $\lambda^m(\mathcal{P})$  up to nonzero factors. We intend to define these operators in a canonical way.

Fix  $\alpha = (\alpha_1, \dots, \alpha_k)$ . Consider  $\mathcal{P} = (P_1, \dots, P_k) \in \mathcal{O}[\alpha]$ . In particular,  $\dim \text{Im } P_j = \alpha_1 + \dots + \alpha_j$ , and  $\dim \text{Indef } P_j = \alpha_1 + \dots + \alpha_{j-1}$

We have the family of the operators

$$\lambda(P_1), \lambda(P_2), \dots, \lambda(P_k) : \Lambda(V) \rightarrow \Lambda(V)$$

defined up to nonzero factors. We have

$$\lambda^m(P_j) \neq 0 \quad \text{iff} \quad \dim \text{Indef } P_j \leq m \leq \dim \text{Im } P_j,$$

and

$$\lambda^{\alpha_1}(P_2) = c_1 \cdot \lambda^{\alpha_1}(P_1),$$

$$\lambda^{\alpha_1+\alpha_2}(P_3) = c_2 \cdot \lambda^{\alpha_1+\alpha_2}(P_2),$$

$$\lambda^{\alpha_1+\alpha_2+\alpha_3}(P_4) = c_3 \cdot \lambda^{\alpha_1+\alpha_2+\alpha_3}(P_3),$$

etc. The linear relation  $P_1$  is a graph of some linear operator  $A$ . Thus the operator  $\lambda(P_1) := \lambda_{\text{cha}}(A)$  is well defined. After this we define the operator  $\lambda(P_2)$  by the condition  $c_1 = 1$ , then we define the operator  $\lambda(P_3)$  by the condition  $c_2 = 1$  etc.

We denote by  $\mathcal{L}^m(\mathcal{P}) : \Lambda^m V \rightarrow \Lambda^m V$  the unique nonzero operator among

$$\lambda^m(P_1), \dots, \lambda^m(P_k).$$

We denote by  $\mathcal{L}(\mathcal{P})$  the collection  $(\mathcal{L}^0(\mathcal{P}), \mathcal{L}^1(\mathcal{P}), \dots, \mathcal{L}^n(\mathcal{P}))$ . Thus we obtain the embedding

$$\mathcal{L} : \text{Hinge}_n \rightarrow \widetilde{\text{Hinge}_n}.$$

REMARK. For the canonical hinge  $\mathcal{P}_\alpha$ , the family of the operators  $\mathcal{L}^m = \mathcal{L}^m(\mathcal{P}_\alpha)$  in the exterior powers  $\Lambda^m(V)$  was described in 2.7.  $\square$

LEMMA 2.11. *For any  $g_1, g_2 \in \mathrm{GL}_n(\mathbb{C})$ ,*

$$\mathcal{L}^m(g_1 \mathcal{P} g_2) = \lambda_{\mathrm{cha}}^m(g_1) \mathcal{L}^m(\mathcal{P}) \lambda_{\mathrm{cha}}^m(g_2).$$

PROOF. Indeed, the multiplication by  $\lambda_{\mathrm{cha}}(g)$  does not change the "gluing conditions"  $c_1 = c_2 = \dots = 1$ .  $\square$

**2.11. Reduced hinge semigroup.** Consider the image  $\mathcal{L}(\mathrm{Hinge}_n)$  of the embedding  $\mathcal{L}$  described in the previous subsection. Denote by  $\overline{\mathrm{Hinge}_n}$  the closure of this image.

The set  $\overline{\mathrm{Hinge}_n}$  is the union of  $2^{n-1}$  affine algebraic varieties  $\overline{\mathcal{L}(\mathcal{O}_\alpha)}$ , the dimension of all these varieties is  $n^2$ .

The set  $\overline{\mathrm{Hinge}_n}$  admits the following explicit description. Let  $\mathcal{R} = (R_1, \dots, R_k)$  be a weak hinge. Consider the family of operators

$$c_j \cdot \lambda(R_j) : \Lambda V \rightarrow \Lambda V,$$

where  $c_j \in \mathbb{C}^*$ . We say that the family is *well glued* if

1. the condition  $\lambda^m(R_j) \neq 0$ ,  $\lambda^m(R_{j+1}) \neq 0$  (or, equivalently,  $\mathrm{Ker}(R_j) = \mathrm{Dom}(R_{j+1})$ , and their dimension is  $m$ ) implies

$$c_j \cdot \lambda^m(R_j) = c_{j+1} \cdot \lambda^m(R_{j+1});$$

2. if  $R_1$  is an operator, then  $c_1 \lambda(R_1) = \lambda_{\mathrm{cha}}(R_1)$ .

The set  $\overline{\mathrm{Hinge}_n}$  coincides with the set of all well-glued families.

Obviously, the multiplication of hinges preserves the glueing condition. This implies the following statement

PROPOSITION 2.12. *The set  $\overline{\mathrm{Hinge}_n}$  is a subsemigroup in the semigroup  $\widetilde{\mathrm{Hinge}}$ .*

**2.12. Representations of the semigroup of hinges.** Recall the construction of irreducible polynomial finite-dimensional representations of  $\mathrm{GL}_n(\mathbb{C})$ .

Consider a collection of integers

$$\nu_1 \geq \nu_2 \geq \dots \geq \nu_n \geq 0. \tag{2.15}$$

We call such collections *signatures*. Denote by  $\pi$  the standard representation of  $\mathrm{GL}_n(\mathbb{C})$  in  $V = \mathbb{C}^n$ . Fix a basis  $e_1, \dots, e_n$  in  $\mathbb{C}^n$ . Denote by  $\xi_s$  the vector

$$\xi_s = e_1 \wedge e_2 \wedge \dots \wedge e_s \in \Lambda^s V.$$

Consider the space

$$\mathfrak{H}_\nu := \bigotimes_{j=1}^n \Lambda^j(V)^{\otimes(\nu_j - \nu_{j+1})}$$

(we assume  $\nu_{n+1} = 0$ ). Consider the representation  $\mathfrak{r}_\nu$  of the group  $\mathrm{GL}_n(\mathbb{C})$  in the space  $\mathfrak{H}_\nu$  given by

$$\mathfrak{r}_\nu(g) = \bigotimes_{j=1}^n \lambda_{\mathrm{cha}}^j(g)^{\otimes(\nu_j - \nu_{j+1})}.$$

Consider also the vector

$$\Xi_\nu := \bigotimes_{j=1}^n \xi_j^{\otimes(\nu_j - \nu_{j+1})} \in \mathfrak{H}_\nu.$$

Denote by  $H_\nu$  the  $\mathrm{GL}_n(\mathbb{C})$ -cyclic span of the vector  $\Xi_\nu$ . We denote by  $\rho_\nu$  the representation of the group  $\mathrm{GL}_n(\mathbb{C})$  in the space  $H_\nu$ .

It is well known (see, for instance [31]), that the representations  $\rho_\nu$  are irreducible and all the polynomial irreducible representations of the group  $\mathrm{GL}_n(\mathbb{C})$  have this form.

Let us define the representations  $\rho_\nu$  of the semigroup  $\widetilde{\mathrm{Hinge}}_n$ . Let

$$\mathcal{A} := (A_0, A_1, A_2, \dots, A_n) \in \widetilde{\mathrm{Hinge}}_n; \quad A_j \in \mathrm{Mat}(\Lambda^j V).$$

We define the operator  $\mathfrak{r}_\nu(\mathcal{A})$  in  $\mathfrak{H}_\nu$  by

$$\mathfrak{r}_\nu(\mathcal{A}) := \bigotimes_{j=1}^n A_j^{\otimes(\nu_j - \nu_{j+1})}.$$

LEMMA 2.13. *The subspace  $H_\nu \subset \mathfrak{H}_\nu$  is invariant with respect to the operators  $\mathfrak{r}_\nu(\mathcal{A})$ .*

PROOF. Assume  $\mathfrak{r}_\nu(\mathcal{A}) \neq 0$ . Then the operator  $\mathfrak{r}_\nu(\mathcal{A})$  depends (up to a scalar factor) only on the hinge  $\mathcal{P}$  lying under  $\mathcal{A}$ . But any hinge can be approximated by elements of  $\mathrm{GL}_n(\mathbb{C})$ .  $\square$

We define the operator

$$\rho_\nu(\mathcal{A})$$

as the restriction of the operator  $\mathfrak{r}_\nu(\mathcal{A})$  to the subspace  $H_\nu$ . Obviously,  $\rho_\nu$  is a linear representation of the semigroup  $\widetilde{\mathrm{Hinge}}_n$ :

$$\rho_\nu(\mathcal{A})\rho_\nu(\mathcal{B}) = \rho_\nu(\mathcal{AB})$$

LEMMA 2.14. *Let  $\mathcal{A}$  be a nondegenerated element of  $\widetilde{\mathrm{Hinge}}_n$  lying over the canonical hinge  $\mathcal{P}_{\alpha_1, \dots, \alpha_k}$ . Then*

$$\rho_\nu(\mathcal{A})\Xi_\nu = c \cdot \Xi_\nu,$$

where  $c \in \mathbb{C}^*$ .

PROOF. Indeed,

$$\lambda^m(\mathcal{P}_{\alpha_1, \dots, \alpha_k})e_1 \wedge \dots \wedge e_m = e_1 \wedge \dots \wedge e_m,$$

see 2.7. This implies the required statement.  $\square$

COROLLARY 2.15. *Let  $\mathcal{A}$  be a nondegenerated element of  $\widetilde{\mathrm{Hinge}}_n$ . Then  $\rho_\nu(\mathcal{A}) \neq 0$ .*

PROOF. Each hinge  $\mathcal{Q}$  is of the form  $g_1 \mathcal{P}_\alpha g_2$ , there  $g_1, g_2 \in \mathrm{GL}(V)$ . Thus,

$$\rho_\nu(\mathcal{A}) = c \cdot \rho_\nu(g_1) \rho_\nu(\mathcal{P}_\alpha) \rho_\nu(g_2).$$

The first and the third factors are invertible and the middle factor is nonzero.  $\square$

**2.13. Extension of reducible representations.** Let  $\zeta$  be a finite-dimensional polynomial representation of  $\mathrm{GL}_n(\mathbb{C})$  in the space  $Z$ . Then

$$\zeta = \oplus \rho_{\nu^{(j)}}; \quad Z = \oplus H_{\nu^{(j)}},$$

where  $\nu^{(1)}, \nu^{(2)} \dots$  are collections of signatures satisfying (2.15), and  $H_\nu$  are the corresponding spaces. We define the representation  $\zeta(\mathcal{A})$  of the semigroup  $\widetilde{\mathrm{Hinge}}_n$  by

$$\zeta(\mathcal{A}) = \oplus \rho_{\nu^{(j)}}(\mathcal{A})$$



**2.14. Identification of  $\text{Hinge}_n^*$  with the De Concini–Procesi compactification of  $\text{GL}_n(\mathbb{C})$ .** Consider an *irreducible* representation  $\rho_\nu$  of the group  $\text{GL}_n(\mathbb{C})$ . Let  $\mathcal{A} \in \widehat{\text{Hinge}}_n$  lies over a hinge  $\mathcal{P}$ . Then the operator  $\rho_\nu(\mathcal{A})$  is determined up to a factor by the underlying hinge  $\mathcal{P}$ . Moreover, it is determined by  $\mathcal{P}$  considered as an element of  $\text{Hinge}_n^*$ . By Corollary 2.15,  $\rho_\nu$  determines the map

$$\text{Hinge}_n^* \rightarrow \mathbb{P}\text{Mat}(H_\nu)$$

It can easily be checked that this map is continuous.

**THEOREM 2.16.** a) *Let  $\rho_\nu$  be an irreducible polynomial representation of  $\text{GL}_n(\mathbb{C})$ . Then the image of  $\text{Hinge}_n^*$  in  $\mathbb{P}\text{Mat}(H_\nu)$  coincides with the projective compactification  $[\text{GL}_n]_{\rho_\nu}$  defined in 0.1.*

b) *If  $\nu_1 > \nu_2 > \dots > \nu_n$ , then the map  $\rho_\nu : \text{Hinge}_n^* \rightarrow \mathbb{P}\text{Mat}(H_\nu)$  is an embedding.*

The statement b) identifies  $\text{Hinge}_n^*$  with the De Concini–Procesi construction, [4].

### 3. Sea Urchin.

In this Section, we show that the hinge language is sufficient for a description of all the projective compactifications of  $\text{GL}_n(\mathbb{C})$ .

**3.1. Meromorphic matrices.** Denote by  $D_\varepsilon$  the disk  $|z| < \varepsilon$  on  $\mathbb{C}$ . Denote by  $\dot{D}_\varepsilon$  the punctured disk  $0 < |z| < \varepsilon$  on  $\mathbb{C}$ .

Denote by  $\text{GL}_N(\mathcal{F})$  the group of germs of holomorphic maps  $D_\varepsilon \rightarrow \text{GL}_N(\mathbb{C})$ . The elements of this group are  $N \times N$  matrices

$$\gamma(z) = \begin{pmatrix} \gamma_{11}(z) & \dots & \gamma_{1N}(z) \\ \vdots & \ddots & \vdots \\ \gamma_{N1}(z) & \dots & \gamma_{NN}(z) \end{pmatrix},$$

where  $\gamma_{ij}(z)$  are functions holomorphic in a neighborhood of 0, and  $\gamma(0)$  is invertible.

We say that a map

$$\gamma : \dot{D}_\varepsilon \rightarrow \text{GL}_N(\mathbb{C})$$

is a *meromorphic family* if all matrix elements  $\gamma_{ij}$  are holomorphic functions in some punctured disk  $\dot{D}_\varepsilon$  with poles or removable singularities at 0, and  $\gamma(z)$  is invertible for  $z$  lying in some punctured disk  $\dot{D}_{\varepsilon'}$ .

We define the order  $\text{ord}(\gamma)$  of the pole of  $\gamma$  as the maximal order of poles at 0 of the matrix elements  $\gamma_{ij}$ . In this definition, we admit a negative order of a pole (a function has a pole of negative order  $-k$  at a point 0, if it has the zero of order  $k$  at 0).

The value  $\text{ord}(\gamma)$  coincides with the minimal  $k$  such that the map

$$z \mapsto z^k \gamma(z)$$

from  $D_\varepsilon$  to  $\text{Mat}(\mathbb{C}^N)$  is holomorphic.

### 3.2. Exponents of meromorphic families.

**LEMMA 3.1.** a) *Any meromorphic family  $\gamma(z) : \dot{D}_\varepsilon \rightarrow \text{GL}_n(\mathbb{C})$  can be represented in the form*

$$\gamma(z) = a(z) \begin{pmatrix} z^{-m_1} & 0 & 0 & \dots \\ 0 & z^{-m_2} & 0 & \dots \\ 0 & 0 & z^{-m_3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} b(z), \quad (3.1)$$

where  $a(z), b(z) \in \mathrm{GL}_n(\mathcal{F})$  and

$$m_1 \geq m_2 \geq \dots \geq m_n \quad (3.2)$$

b) Consider the  $j$ -th exterior power  $\lambda_{\mathrm{cha}}^j(\gamma(z))$  of the matrix  $\gamma(z)$ . Then

$$\mathrm{ord} \lambda_{\mathrm{cha}}^j(\gamma(z)) = m_1 + m_2 + \dots + m_j$$

In particular, the numbers  $m_j$  are uniquely determined by the function  $\gamma(z)$ .

PROOF. a) It is sufficient to apply the Gauss elimination algorithm.

b) Obvious.  $\square$

We say that the numbers  $m_i$  are the *exponents* of the meromorphic family  $\gamma(z)$ . We fix the notation

$$\mathbf{m} := (m_1, \dots, m_n)$$

for the exponents. We also define the numbers

$$k_1 > k_2 > \dots > k_\tau$$

to be all the pairwise different exponents  $m_i$ . We denote by  $\alpha_j$  the number of copies  $k_j$  in the collection  $(m_1, m_2, \dots, m_n)$ . We say that  $\alpha_j, k_j$  are the *numbers associated with  $m_i$* .

Starting this place, the sense of numbers  $m_i, \alpha_j, k_j$  is fixed.

**3.3. Limits of meromorphic families in the  $\mathrm{Hinge}_n$ .** Let  $\gamma(z)$  be a meromorphic family of  $n \times n$  matrices. Let  $m_1, \dots, m_n$  be its exponents, and let  $k_1, \dots, k_\tau, \alpha_1, \dots, \alpha_\tau$  be the associated numbers.

PROPOSITION 3.2. a) For each  $j$  the following limit in  $\mathrm{Gr}_n$

$$P_j := \lim_{z \rightarrow 0} z^{k_j} \mathrm{graph}(\gamma(z)) \quad (3.3)$$

exists, and  $\mathrm{rk} P_j = \alpha_j$ .

b) The collection  $\mathcal{P}^\gamma := (P_1, \dots, P_\tau)$  is a hinge.

REMARK. In 2.4, we defined the convergence of sequences  $g_j \in \mathrm{PGL}_n(\mathbb{C})$  to elements of  $\mathrm{Hinge}_n^*$ . It is impossible to define a convergence of sequences  $g_j \in \mathrm{GL}_n(\mathbb{C})$  to points of  $\mathrm{Hinge}_n$ . But in Proposition 3.2, we consider holomorphic families instead of sequences.

PROOF. Let  $\gamma(z)$  be the diagonal matrix with the eigenvalues  $z^{-m_1}, \dots, z^{-m_n}$ . In this case, the limit of  $\gamma(z)$  is the canonical hinge  $\mathcal{P}_\alpha$ . For a general family (3.1), the limit is

$$a(0)\mathcal{P}_\alpha b(0)$$

**3.4. Limits of meromorphic families in  $\overline{\mathrm{Hinge}_n}$ .** Let  $\gamma$  be a meromorphic family, let  $m_j$  be its exponents.

PROPOSITION 3.3 a) For each  $j = 0, 1, \dots, n$  there exists the nonzero limit

$$\mathcal{L}^j = \lim_{z \rightarrow 0} z^{m_1 + \dots + m_j} \lambda_{\mathrm{cha}}^j(\gamma(z)) \quad (3.4)$$

in  $\mathrm{Mat}(\Lambda^j V)$ .

b) The collection  $\mathcal{L}_j$  coincides with the collection  $\mathcal{L}^j(\mathcal{P}^\gamma)$ , where  $\mathcal{P}^\gamma$  is the hinge constructed in Proposition 3.2 and the operators  $\mathcal{L}^j(\mathcal{P})$  were described in 2.10.

EXAMPLE. Consider the diagonal matrix  $\delta_{\mathbf{m}}(z)$  with the eigenvalues  $z^{-m_1}, \dots, z^{-m_n}$ . Then  $\mathcal{P}^{\delta_{\mathbf{m}}}$  is the canonical hinge  $\mathcal{P}_\alpha$ . The operator  $\lambda_{\text{cha}}^j(\delta_{\mathbf{m}}(z))$  is the diagonal operator in the basis  $e_{p_1} \wedge \dots \wedge e_{p_j}$ . The eigenvalues are  $z^{-m_{p_1} - \dots - m_{p_j}}$ . The maximal absolute value of the eigenvalues is  $|z|^{-m_1 - \dots - m_j}$ .

The eigenspace  $W_j \subset \Lambda^j V$  corresponding to the maximal eigenvalue is spanned by the vectors

$$e_1 \wedge e_2 \wedge \dots \wedge e_u \wedge e_{q_1} \wedge \dots \wedge e_{q_{j-u}}, \quad (3.5)$$

where  $u$  is the largest  $i$  such that  $m_i > m_j$  and  $m_{q_1} = \dots = m_{q_{j-u}} = m_j$ . Obviously,

$$\mathcal{L}^j = \lim_{z \rightarrow 0} z^{m_1 + \dots + m_j} \lambda_{\text{cha}}^j(\delta_{\mathbf{m}}(z))$$

is the identical operator on the subspace spanned by the vectors (3.5), and  $\mathcal{L}^j$  annihilate all other basic vectors. Thus,  $\mathcal{L}^j$  coincides with the operator  $\mathcal{L}^j(\mathcal{P}_\alpha)$  described in 2.7.  $\square$

PROOF OF PROPOSITION 3.3. Let us represent  $\gamma(z)$  in the form (3.1). Then, by the example given above,

$$\mathcal{P}^\gamma = a(0)\mathcal{P}_\alpha b(0); \quad \mathcal{L}^j = \lambda_{\text{cha}}^j(a(0))\mathcal{L}^j(\mathcal{P}_\alpha)\lambda_{\text{cha}}^j(b(0)).$$

Now we apply Lemma 2.11.  $\square$

**3.5. Limits of meromorphic families in irreducible representations.** Let  $\rho_\nu$  be an irreducible holomorphic representation of  $\text{GL}_n(\mathbb{C})$ . We extend  $\rho_\nu$  to the representation of  $\widetilde{\text{Hinge}}_n$  by the procedure 2.12.

LEMMA 3.4. Let  $\gamma$  be a meromorphic family of  $n \times n$  matrices, let  $m_i$  be its exponents. Let  $\mathcal{L} : \widetilde{\text{Hinge}}_n \rightarrow \widetilde{\text{Hinge}}_n$  be the embedding defined in 2.10.

a) There exists the nonzero limit

$$[\rho_\nu(\gamma)] := \lim_{z \rightarrow 0} z^{\sum m_i \nu_i} \rho_\nu(\gamma(z)). \quad (3.6)$$

b) This limit coincides with the operator

$$\rho_\nu(\mathcal{L}(\mathcal{P}^\gamma)).$$

PROOF. Obvious.  $\square$

**3.6. Change of the variable  $z$  in meromorphic families.** We intend to consider a limit of a meromorphic curve independently on its parametrization.

a) If we change the variable  $z$  by the formula

$$z \mapsto z + c_2 z^2 + c_3 z^3 + \dots,$$

then nothing changes. The exponents  $m_j$ , the hinge  $\mathcal{P}^\gamma$ , the limit (3.4) in  $\bigoplus \text{Mat}(\Lambda^j V)$ , and limit (3.6) in  $\text{Mat}(H_\nu)$  remain the same.

b) If we change the variable  $z$  by the formula

$$z \mapsto z^p,$$

then the exponents  $m_j$  are replaced by  $pm_j$ . All other data ( i.e.,  $\mathcal{P}^\gamma$  and  $\mathcal{L}^j$ ) remain the same as above.

c) Let us change the variable  $z$  by

$$c \mapsto cz, \quad \text{where } c \in \mathbb{C}^*.$$

Then  $m_j$  do not change. The hinge  $\mathcal{P}^\gamma$  transforms by the rule

$$(P_1, \dots, P_\tau) \mapsto (c^{-k_1} P_1, \dots, c^{-k_\tau} P_\tau).$$

The collection  $\mathcal{L}(\mathcal{P}^\gamma) = (\mathcal{L}^0(\mathcal{P}^\gamma), \dots, \mathcal{L}^n(\mathcal{P}^\gamma)) \in \widetilde{\text{Hinge}}_n$  transforms by the rule

$$(\mathcal{L}^0, \mathcal{L}^1, \mathcal{L}^2, \dots) \mapsto (c^0 \cdot \mathcal{L}^0, c^{-m_1} \mathcal{L}^1, c^{-m_1-m_2} \mathcal{L}^2, \dots).$$

The operators  $[\rho_\nu(\gamma)]$  transform by the rule

$$[\rho_\nu(\gamma)] \mapsto c^{-\sum m_j \nu_j} [\rho_\nu(\gamma)].$$

**3.7. Sea urchin.** We define the *sea urchin*  $\mathfrak{S}\mathfrak{U}_n$  as the union of  $\text{GL}_n(\mathbb{C})$  and all the spikes  $\mathfrak{sp}[\mathbf{m}]$ .

The *spikes*  $\mathfrak{sp}[\mathbf{m}]$  are enumerated by the collections of integers

$$\mathbf{m} : m_1 \geq m_2 \geq \dots \geq m_n$$

such that  $m_1, \dots, m_n$  have no common divisor. Let  $k_j, \alpha_j$  be the associated numbers (see 3.2). Points of the spike  $\mathfrak{sp}[\mathbf{m}]$  are hinges  $\mathcal{P} \in \mathcal{O}[\alpha_1, \dots, \alpha_\tau]$  defined up to the equivalence

$$(P_1, \dots, P_\tau) \sim (c^{k_1} P_1, \dots, c^{k_\tau} P_\tau); \quad c \in \mathbb{C}^*. \quad (3.7)$$

REMARK.  $\dim \text{GL}_n = n^2$ , and the dimension of all spikes is  $n^2 - 1$ .  $\square$

**3.8. Limits of meromorphic curves in sea urchin.** Let  $\gamma$  be a meromorphic curve in  $\text{GL}_n(\mathbb{C})$ . Let  $\hat{m}_1, \dots, \hat{m}_n$  be its exponents, let  $u$  be the greatest (positive) common divisor of the numbers  $\hat{m}_j$ . Then we define the collection  $\mathbf{m}^\gamma$  by

$$m_j = m_j^\gamma = \hat{m}_j / u$$

The limit  $\mathcal{P}^\gamma$  of the curve  $\gamma$  in the spike  $\mathfrak{sp}[\mathbf{m}]$  is defined by Proposition 3.2.

By 3.6, the collection  $\mathbf{m}^\gamma$  and the limit of the curve  $\gamma$  in  $\mathfrak{sp}[\mathbf{m}]$  do not depend on the parametrization of the curve  $\gamma$ .

**3.9. Projective compactifications of  $\text{GL}_n(\mathbb{C})$ .** Let  $\rho_{\nu^{(1)}}, \dots, \rho_{\nu^{(\sigma)}}$  be an irreducible holomorphic representations of  $\text{GL}_n(\mathbb{C})$ , where

$$\nu^{(l)} : \nu_1^{(l)} \geq \dots \geq \nu_n^{(l)}$$

are signatures (see 2.11). Let  $H_{\nu^{(l)}}$  be the spaces of the representations  $\rho_{\nu^{(l)}}$ . Let

$$\zeta := \bigoplus_{l=1}^{\sigma} \rho_{\nu^{(l)}}; \quad Z = \bigoplus_{l=1}^{\sigma} H_{\nu^{(l)}}.$$

We have the maps

$$\text{GL}_n(\mathbb{C}) \xrightarrow{\zeta} \text{Mat}(Z) \setminus 0 \rightarrow \mathbb{P}\text{Mat}(Z). \quad (3.8)$$

Denote by  $[\mathrm{GL}_n]_\zeta$  the closure of the image of  $\mathrm{GL}_n(\mathbb{C})$  in  $\mathbb{P}\mathrm{Mat}(Z)$ . The spaces  $\mathbb{P}\mathrm{Mat}(Z)$  are called *projective compactifications* of  $\mathrm{GL}_n(\mathbb{C})$ .

**3.9. The canonical maps**  $\pi_\zeta : \mathfrak{S}\mathfrak{U}_n \rightarrow \mathbb{P}\mathrm{Mat}(Z)$ . We must extend the map (3.8) to the spikes  $\mathfrak{sp}[\mathbf{m}]$  of the sea urchin  $\mathfrak{S}\mathfrak{U}_n$ .

Let us fix a spike  $\mathfrak{sp}[\mathbf{m}]$ . Consider the numbers

$$v^{(l)} := \sum_j m_j \nu_j^{(l)}.$$

Let

$$v := \max_l v^{(l)}.$$

For a hinge  $\mathcal{P} \in \mathfrak{sp}[\mathbf{m}]$ , we define the operator

$$\zeta(\mathbf{m}; \mathcal{P}) = \bigoplus B_l(\mathbf{m}, \mathcal{P}) : \bigoplus H_{\nu^{(l)}} \rightarrow \bigoplus H_{\nu^{(l)}},$$

where the operators

$$B_l = B_l(\mathbf{m}; \mathcal{P}) : H_{\nu^{(l)}} \rightarrow H_{\nu^{(l)}}$$

are given by

$$B_l = \begin{cases} \rho_{\nu^{(l)}}(\mathcal{L}(\mathcal{P}^\gamma)), & \text{if } v^{(l)} = v \\ 0, & \text{if } v^{(l)} < v \end{cases}$$

REMARK. The hinge  $\mathcal{P} \in \mathfrak{sp}[\mathbf{m}]$  is defined up to the equivalence (3.7). Hence the operators  $[\rho_{\nu^{(l)}}(\mathcal{P}^\gamma)]$  are defined up to the factors  $c^{\sum_i m_j \nu_j^{(l)}} = c^{v^{(l)}}$ . But for all nonzero  $B_l$ , these factors coincide, and hence  $\zeta(\mathbf{m}; \mathcal{P})$  is a nonzero operator defined up to a factor  $c^v$ .  $\square$

THEOREM 3.5 *The image of the sea urchin under the map  $(\mathbf{m}, \mathcal{P}) \mapsto \zeta(\mathbf{m}, \mathcal{P})$  coincides with  $[\mathrm{GL}_n]_\zeta$ .*

LEMMA 3.6. *Let  $z \mapsto \gamma(z)$  be a meromorphic family. Then the limit of  $\zeta(\gamma(z))$  in  $[\mathrm{GL}_n]_\zeta$  exists and coincides with  $\zeta(\mathbf{m}^\gamma, \mathcal{P}^\gamma)$ .*

PROOF OF LEMMA 3.6.

$$\mathrm{ord}_{\nu^{(l)}}(\gamma(z)) = \sum_i \nu_i^{(l)} m_i = v^{(l)}$$

Thus our limit is

$$\lim_{z \rightarrow 0} z^v \zeta(\gamma(z)) = \lim_{z \rightarrow 0} z^v \bigoplus_l \rho_{\nu^{(l)}}(\gamma(z))$$

and Lemma 3.4 implies the required result.  $\square$

PROOF OF THEOREM 3.5. The set  $[\mathrm{GL}_n]_\zeta \setminus \mathrm{GL}_n(\mathbb{C})$  is a subvariety of the projective variety  $[\mathrm{GL}_n]_\zeta$ . Thus any point of this set can be achieved along an algebraic curve  $\gamma(z) \subset \mathrm{GL}_n(\mathbb{C})$ . By Lemma 3.6, this point is contained in the image of the sea urchin.

Conversely, each point  $(\mathbf{m}, \mathcal{P})$  of the sea urchin can be achieved along an algebraic curve  $\gamma(z)$ . By Lemma 3.6, the image of the point  $(\mathbf{m}, \mathcal{P})$  can be achieved along the curve  $\zeta(\gamma(z))$  and thus  $\zeta(\mathbf{m}, \mathcal{P}) \in [\mathrm{GL}_n]_\zeta$ .  $\square$

REMARK. Let us give a description of the sea urchin in formal terms. The spikes are homogeneous spaces  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) / G_{\mathbf{m}}$ , where the family of subgroups  $G_{\mathbf{m}}$  can be obtained by the following algorithm.

Consider two opposite parabolic subgroups  $P_+$  and  $P_-$  in  $\mathrm{GL}_n(\mathbb{C})$ . Denote by  $N_+$  and  $N_-$  the unipotent radicals in  $P_+$ ,  $P_-$ . Denote by  $Q$  the Levi subgroup ( $Q \subset P_+$ ,  $Q \subset P_-$ ).

Denote by  $Z$  the center of the Levi subgroup. Let  $S$  be an one-parametric subgroup in  $Z$ . We say that  $S$  is positive if all root lying in  $N_+$  are positive on the generator of  $S$ . Now we consider the subgroup

$$G^S \subset \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$$

consisting of all pairs

$$(g_1, g_2) = (n_- q_1, q_2 n_+)$$

where  $n_- \in N_-$ ,  $n_+ \in N_+$ ,  $q_1, q_2 \in Q$  and  $q_1^{-1} q_2 \in S$ . Then the family of the subgroups  $G^S$  coincides with the family  $G_{\mathbf{m}}$ .

This remark allows to extend the sea urchin construction to an arbitrary complex semisimple group. For all classical groups the De Concini–Procesi compactification can be described on the hinge language (see [19], [22]) and the sea urchins also can easily be described in these terms.  $\square$

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